

# Weak-Pseudo-Hermiticity of Non-Hermitian Hamiltonians with Position-Dependent Mass

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## Abstract

We extend the definition of  $\eta$ -weak-pseudo-Hermiticity to the class of potentials endowed with position-dependent mass. The construction of non-Hermitian Hamiltonians through some generating function are obtained. Special cases of potentials are thus deduced.

Keywords :  $\eta$ -weak-pseudo-Hermiticity; Non-Hermitian Hamiltonians;  
 $\mathcal{PT}$ -symmetry; Effective mass.

PACS : 03.65.Ca; 03.65.Fd; 03.65.Ge

## 1 Introduction

The Hamiltonians are called  $\mathcal{PT}$ -invariant if they are invariant under a joint transformation of parity  $\mathcal{P}$  and time-reversal  $\mathcal{T}$  [1-8]. A conjecture due to Bender and Boettcher [1] has relaxed  $\mathcal{PT}$ -symmetry as a necessary condition for the reality of the spectrum. Here, the Hermiticity assumption  $\mathcal{H} = \mathcal{H}^\dagger$  is replaced by the  $\mathcal{PT}$ -symmetric one; i.e.  $[\mathcal{PT}, \mathcal{H}] = 0$ , where  $\mathcal{P}$  denotes the parity operator (space reflection) and has as effects :  $x \rightarrow -x$ ,  $p \rightarrow -p$  and  $\mathcal{T}$  mimics the time-reversal and has as effects :  $x \rightarrow x$ ,  $p \rightarrow -p$ , and  $i \rightarrow -i$ . Note that  $\mathcal{T}$  changes the sign of  $i$  because it preserves the fundamental commutation relation of the quantum mechanics known as the Heisenberg algebra, i.e.  $[x, p] = i\hbar$  [1-3].

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According to Mostafazadeh [9-12], the basic mathematical structure underlying the properties of  $\mathcal{PT}$ -symmetry is explored and can now be found to be connected to the concept of a pseudo-Hermiticity. The pseudo-Hermiticity has been found to be a more general concept than those of Hermiticity and  $\mathcal{PT}$ -symmetry. As a consequence of this, the reality of the bound-state eigenvalues can be associated with it.

In terms of these settings, a Hamiltonian  $\mathcal{H}$  is called pseudo-Hermitian if it obeys to [9,11]

$$\mathcal{H}^\dagger = \eta \mathcal{H} \eta^{-1}, \quad (1)$$

where  $\eta$  is a Hermitian invertible linear operator and a dagger ( $^\dagger$ ) stands for the adjoint of the corresponding operator. A non-Hermitian Hamiltonian has a real spectrum if and only if it is pseudo-Hermitian with respect to a linear Hermitian automorphism [10], and may be factored as

$$\eta = \mathcal{D}^\dagger \mathcal{D}, \quad (2)$$

where  $\mathcal{D} : \mathfrak{H} \rightarrow \mathfrak{H}$  is a linear automorphism ( $\mathfrak{H}$  is the Hilbert space). Note that choosing  $\eta = 1$  reduces the assumption (1) to the Hermiticity of the Hamiltonian.

On the other hand, Bagchi and Quesne [13] have established that the twin concepts of pseudo-Hermiticity and weak-pseudo-Hermiticity are complementary to one another. In the pseudo-Hermiticity case,  $\eta$  can be written as a first-order differential operator and may be anti-Hermitian, while in the weak-pseudo-Hermitian case,  $\eta$  is a second-order differential operator and must be necessarily Hermitian.

The quantum mechanical systems with position-dependent mass have attracted, in recent years, much attention on behalf of physicists [15-20]. The effective mass Schrödinger equation was first introduced by BenDaniel and Duke in order to explain the behaviors of electrons in semi-conductors [15]. It also have many applications in the fields of materials science and condensed matter physics [20,21].

In the present paper, a class of non-Hermitian Hamiltonians, known in the literature, as well as their accompanying ground-state wavefunctions are generated as a by-product of the generalized  $\eta$ -weak-pseudo-Hermiticity endowed with position-dependent mass. Here our primary

concern is to point out that, being different from the realization of Ref.[13] considering therein  $A(x)$  as a pure imaginary function, there is no inconsistency if a shift on the momentum  $p$  of the type  $p \rightarrow p - \frac{A(x)}{U(x)}$  is used, where  $A(x)$  and  $U(x) (\neq 0)$  are, respectively, complex- and real-valued functions. It opens a way towards the construction of non-Hermitian Hamiltonians (not necessarily  $\mathcal{PT}$ -symmetric). On these settings, Eq.(2) becomes  $\eta \rightarrow \tilde{\eta} = \tilde{\mathcal{D}}^\dagger \tilde{\mathcal{D}}$ . Such operator, i.e.  $\tilde{\mathcal{D}}$ , may be looked upon as a gauge-transformed version of  $\mathcal{D}$ , depending essentially on the function  $A(x)$ . Consequently, it is found that the wavefunction is also subjected to a gauge transformation of the type  $\psi(x) \rightarrow \xi(x) = \Lambda(x) \psi(x)$  where  $\Lambda(x) = \exp \left[ i \int^x dy \frac{A(y)}{U(y)} \right]$ .

## 2 Generalized pseudo-Hermitian Hamiltonians

The general form of the Hamiltonian introduced by von Roos [16] for the spatially varying mass  $M(x) = m_0 m(x)$  reads

$$\mathcal{H} = \frac{1}{4} \left[ m^\alpha(x) p m^\beta(x) p m^\gamma(x) + m^\gamma(x) p m^\beta(x) p m^\alpha(x) \right] + V(x), \quad (3)$$

where the constraint  $\alpha + \beta + \gamma = -1$  holds and  $V(x) = V_{\text{Re}}(x) + iV_{\text{Im}}(x)$  is a complex-valued potential. Here,  $p (= -i\frac{d}{dx})$  is a momentum with  $\hbar = m_0 = 1$ , and  $m(x)$  is dimensionless real-valued mass function.

Using the restricted Hamiltonian from the  $\alpha = \gamma = 0$  and  $\beta = -1$  constraints, the Hamiltonian (3) becomes

$$\mathcal{H} = p U^2(x) p + V(x), \quad (4)$$

with  $U^2(x) = \frac{1}{2m(x)}$ . The shift on the momentum  $p$  in the manner

$$p \rightarrow p - \frac{A(x)}{U(x)}, \quad (5)$$

where  $A : \mathbb{R} \rightarrow \mathbb{C}$  is a complex-valued function, allows to bring the Hamiltonian of Eq.(4) in the form

$$\mathcal{H} \rightarrow \mathcal{H}' = \left[ p - \frac{A(x)}{U(x)} \right] U^2(x) \left[ p - \frac{A(x)}{U(x)} \right] + V(x). \quad (6)$$

In Ref.[11], it was showed that for every anti-pseudo-Hermitian Hamiltonian  $\mathcal{H}$ , there is an antilinear operator  $\tau$  fulfilling the condition

$$\mathcal{H}^\dagger = \tau \mathcal{H} \tau^{-1}. \quad (7)$$

Let us extend the proof of Ref.[12] to our Hamiltonian (6). To this end,  $\tau$  should be constructed suitably. According to Mostafazadeh [12],  $\tau = \mathcal{T} e^{i\alpha(x)}$  is the product of linear and antilinear operators, and  $\alpha : \mathbb{R} \rightarrow \mathbb{C}$  is a complex-valued function. Therefore, the Hermiticity of  $\tau$  is established straightforwardly

$$\tau^\dagger = e^{-i\alpha^*(x)} \mathcal{T}^\dagger = e^{-i\alpha^*(x)} \mathcal{T} = \mathcal{T} e^{i\alpha(x)} = \tau, \quad (8)$$

where the identities  $\mathcal{T}^\dagger = \mathcal{T}$  and  $\mathcal{T} f(x) \mathcal{T} = f^*(x)$  are used and  $f : \mathbb{R} \rightarrow \mathbb{C}$ .

According to Mostafazadeh in Ref.[12], the function  $\alpha(x)$  can be gen-

eralized to  $\alpha(x) = -2 \int^x dy \frac{A(y)}{U(y)}$ , therefore

$$\begin{aligned}
\tau \mathcal{H}' \tau^{-1} &= \mathcal{T} e^{i\alpha(x)} \left[ p - \frac{A(x)}{U(x)} \right] U^2(x) \left[ p - \frac{A(x)}{U(x)} \right] e^{-i\alpha(x)} \mathcal{T} \\
&\quad + \mathcal{T} e^{i\alpha(x)} V(x) e^{-i\alpha(x)} \mathcal{T} \\
&= \mathcal{T} \left[ p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] e^{i\alpha(x)} U^2(x) e^{-i\alpha(x)} \left[ p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] \mathcal{T} \\
&\quad + V^*(x) \\
&= \mathcal{T} \left[ p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] U^2(x) \left[ p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] \mathcal{T} + V^*(x) \\
&= \mathcal{T} \left[ p + \frac{A(x)}{U(x)} \right] U^2(x) \left[ p + \frac{A(x)}{U(x)} \right] \mathcal{T} + V^*(x) \\
&= \left[ -p + \frac{A^*(x)}{U(x)} \right] U^2(x) \left[ -p + \frac{A^*(x)}{U(x)} \right] + V^*(x) \\
&= \left[ p - \frac{A^*(x)}{U(x)} \right] U^2(x) \left[ p - \frac{A^*(x)}{U(x)} \right] + V^*(x) \\
&= \mathcal{H}'^\dagger,
\end{aligned} \tag{9}$$

where for every differential function  $\alpha(x)$ , the following identity holds  $e^{-i\alpha(x)} p e^{i\alpha(x)} = p + \partial_x \alpha(x)$  while the position  $x$  commutes with  $e^{i\alpha(x)}$  and remains unaffected under a last transformation; i.e.  $e^{-i\alpha(x)} x e^{i\alpha(x)} = x$ . Here we note that for every function  $f : \mathbb{R} \rightarrow \mathbb{C}$ , the identity  $\mathcal{T} f(x, p) \mathcal{T} = f^*(x, -p)$  is used.

In the other hand, and according to Ref.[11], it was checked that  $\mathcal{PT}$ -symmetry ( $[\mathcal{PT}, \mathcal{H}] = 0$ ) and anti-pseudo-Hermiticity operator  $\tau$  imply pseudo-Hermiticity of  $\mathcal{H}$  with the respect of a linear Hermitian automorphism  $\eta : \mathfrak{H} \rightarrow \mathfrak{H}$  according to

$$\eta = \tau \mathcal{PT}, \tag{10}$$

and it turns out that the choice of  $\eta$  is not unique. As was made for  $\tau$ , let us generalize  $\eta$  according to

$$\eta = \exp \left[ 2i \int^x dy \frac{A^*(y)}{U(y)} \right] \mathcal{P}, \tag{11}$$

then the Hermiticity of  $\eta$  is established straightforwardly

$$\begin{aligned}
\eta^\dagger &= \mathcal{P} \exp \left[ -2i \int^x dy \frac{A(y)}{U(y)} \right] = \exp \left[ -2i \int^{-x} dy \frac{A(y)}{U(y)} \right] \mathcal{P} \\
&= \exp \left[ 2i \int^{-x} d(-y) \frac{A(y)}{U(y)} \right] \mathcal{P} = \exp \left[ 2i \int^x dy \frac{A(-y)}{U(-y)} \right] \mathcal{P} \\
&= \exp \left[ 2 \int^x dy \frac{i \operatorname{Re} A(-y) - \operatorname{Im} A(-y)}{U(-y)} \right] \mathcal{P} \\
&= \exp \left[ 2 \int^x dy \frac{i \operatorname{Re} A(y) + \operatorname{Im} A(y)}{U(y)} \right] \mathcal{P} \\
&= \exp \left[ 2i \int^x dy \frac{\operatorname{Re} A(y) - i \operatorname{Im} A(y)}{U(y)} \right] \mathcal{P} \\
&= \exp \left[ 2i \int^x dy \frac{A^*(y)}{U(y)} \right] \mathcal{P} \\
&= \eta,
\end{aligned} \tag{12}$$

where we use  $\mathcal{P}^\dagger = \mathcal{P}$  and, for every function  $f : \mathbb{R} \rightarrow \mathbb{C}$ , the following identity holds  $\mathcal{P}f(x)\mathcal{P} = f(-x)$ . In Eq.(12), the real and imaginary parts of  $A(x)$  are, respectively, even and odd functions; i.e.  $\operatorname{Re} A(-x) = \operatorname{Re} A(x)$ ,  $\operatorname{Im} A(-x) = -\operatorname{Im} A(x)$  and  $U(x)$  must be an even function, i.e.  $U(x) = U(-x)$ .

In summary, the  $\mathcal{PT}$ -symmetry and anti-pseudo-Hermiticity with respect to  $\tau$  imply pseudo-Hermiticity with respect to  $\tau\mathcal{PT}$  and which coincides with the  $\eta$  operator [11]. Therefore, it is obvious that the (weak-) pseudo-Hermiticity as defined in Eq.(10) adapts very well to the problems relating with position-dependent effective mass.

### 3 The generalized weak-pseudo-Hermiticity generators

As  $\eta$  is weak-pseudo-Hermitian, then the operators  $\mathcal{D}$  and  $\mathcal{D}^\dagger$  are connected to the first-order differential operator through [14]

$$\begin{aligned}\mathcal{D} &= U(x) \partial_x + \phi(x), \\ &= iU(x) p + \phi(x),\end{aligned}\tag{13.a}$$

$$\begin{aligned}\mathcal{D}^\dagger &= -\partial_x U(x) + \phi^*(x), \\ &= -ipU(x) + \phi^*(x),\end{aligned}\tag{13.b}$$

where we have used the abbreviation  $\partial_x = \frac{d}{dx}$ . Here  $\phi : \mathbb{R} \rightarrow \mathbb{C}$  is a complex-valued function. It is obvious that the operator  $\mathcal{D}$  becomes, under transformation (5),

$$\begin{aligned}\tilde{\mathcal{D}} &= iU(x) \left[ p - \frac{A(x)}{U(x)} \right] + \phi(x), \\ &= iU(x) p - iA(x) + \phi(x).\end{aligned}\tag{14}$$

Therefore, the operator  $\tilde{\mathcal{D}}$  may be looked upon as a gauge-transformed version of  $\mathcal{D}$ , depending on  $A(x)$  such that  $\tilde{\mathcal{D}} = \mathcal{D} - iA(x)$ . In terms of these,  $\tilde{\eta}$  becomes

$$\begin{aligned}\tilde{\eta} &= \tilde{\mathcal{D}}^\dagger \tilde{\mathcal{D}} \\ &= [\mathcal{D}^\dagger + iA^*(x)] [\mathcal{D} - iA(x)] \\ &= \mathcal{D}^\dagger \mathcal{D} - i\mathcal{D}^\dagger A(x) + iA^*(x) \mathcal{D} + A^*(x) A(x),\end{aligned}\tag{15}$$

and taking into account that  $\phi(x) = f(x) + ig(x)$  and  $A(x) = a(x) + ib(x)$ , (15) can be recast as

$$\begin{aligned}\tilde{\eta} &= \mathcal{D}^\dagger \mathcal{D} + 2iU(x) a(x) \partial_x + i[U(x) A(x)]' - i\phi^*(x) A(x) \\ &\quad + i\phi(x) A^*(x) + |A(x)|^2,\end{aligned}\tag{16}$$

where prime denotes derivative with respect to  $x$ . At this point, let us now

evaluate  $\eta$  appearing in Eq.(16) using Eq.(13), we obtain

$$\begin{aligned}\mathcal{D}^\dagger \mathcal{D} &= [-\partial_x U(x) + \phi(x)] [U(x) \partial_x + \phi(x)] \\ &= -U^2(x) \partial_x^2 - 2U(x) [U'(x) + ig(x)] \partial_x + |\phi(x)|^2 \\ &\quad - [U(x) \phi(x)]',\end{aligned}\tag{17}$$

Combining Eq.(17) with Eq.(16), we obtain a second-order differential operator of  $\tilde{\eta}$

$$\tilde{\eta} = -U^2(x) \partial_x^2 - 2\mathcal{K}(x) \partial_x + \mathcal{L}(x),\tag{18}$$

where  $\mathcal{K}(x)$  and  $\mathcal{L}(x)$  are defined as

$$\mathcal{K}(x) = U(x) U'(x) + iU(x) g(x) - iU(x) a(x),\tag{19.a}$$

$$\begin{aligned}\mathcal{L}(x) &= |\phi(x)|^2 + |A(x)|^2 - [U(x) \phi(x)]' + i[U(x) A(x)]' \\ &\quad - i\phi^*(x) A(x) + i\phi(x) A^*(x).\end{aligned}\tag{19.b}$$

One can easily check that  $\tilde{\eta}$  given in Eq.(18) is, indeed, Hermitian since it is written in the form  $\tilde{\eta} = \tilde{\mathcal{D}}^\dagger \tilde{\mathcal{D}}$ . On the other hand, taking into account  $p = -i\partial_x$ , the Hamiltonian of Eq.(6) may be expressed as

$$\mathcal{H}' = -U^2(x) \partial_x^2 - 2\mathcal{M}_1(x) \partial_x + \mathcal{N}_1(x) + V(x),\tag{20}$$

where, by definition

$$\mathcal{M}_1(x) = U(x) U'(x) - iU(x) A(x),\tag{21.a}$$

$$\mathcal{N}_1(x) = i[U(x) A(x)]' + A^2(x).\tag{21.b}$$

The adjoint of the Hamiltonian (20) reads as

$$\mathcal{H}'^\dagger = -U^2(x) \partial_x^2 - 2\mathcal{M}_2(x) \partial_x + \mathcal{N}_2(x) + V^*(x),\tag{22}$$

with

$$\mathcal{M}_2(x) = U(x) U'(x) - iU(x) A^*(x),\tag{23.a}$$

$$\mathcal{N}_2(x) = i[U(x) A^*(x)]' + A^{*2}(x).\tag{23.b}$$



It should be noted that  $\mathcal{D}$  and  $\mathcal{D}^\dagger$  are two intertwining operators, therefore, the defining condition (1) may be expressed as  $\eta\mathcal{H} = \mathcal{H}^\dagger\eta$ . Thereupon, a generalization beyond the pair  $\tilde{\eta}$  and  $\mathcal{H}'$  is straightforward, given

$$\tilde{\eta}\mathcal{H}' = \mathcal{H}'^\dagger\tilde{\eta}. \quad (24)$$

Letting both sides of (24) act on every function, e.g. on a wavefunction. Using Eqs.(18), (20), (22) and comparing between their varying differential coefficients, we can easily recognized from the coefficients corresponding to the third derivative that  $A(x)$  must be real function, i.e.  $b(x) = 0$ .

By comparing both coefficients corresponding to the second derivative, one deduces the expression connecting the potential to its conjugate through

$$V(x) = V^*(x) - 4iU(x)g'(x). \quad (25)$$

On the other hand, the coefficients corresponding to the first derivative give the shape of the potential

$$V^{*'}(x) = 2f(x)f'(x) - 2g(x)g'(x) - [U(x)f(x)]'' + 2i[U(x)g'(x)]', \quad (26)$$

and by integrating Eq.(26) taking into account its conjugate, we get

$$\begin{aligned} V(x) &\equiv V_{\text{Re}}(x) + iV_{\text{Im}}(x) \\ &= f^2(x) - g^2(x) - [U(x)f(x)]' - 2iU(x)g'(x) + \delta, \end{aligned} \quad (27)$$

with  $\delta$  is a constant of integration. It is obvious that both imaginary parts of Eqs.(25) and (27) coincide.

The last remaining coefficients correspond to the null derivative and give the following pure-imaginary expression

$$\begin{aligned} &-4U(x)f(x)f'(x)g'(x) - 4U(x)f^2(x)g'(x) + 4U^2(x)f'(x)g'(x) \\ &+ 4U(x)U'(x)f'(x)g(x) + 4U(x)U'(x)f(x)g'(x) + 2U^2(x)f''(x)g(x) \\ &+ 3U^2(x)U'(x)g''(x) + 2U(x)U''(x)f(x)g(x) - U^2(x)U''(x)g'(x) \\ &- 2U(x)U'(x)U''(x)g(x) + U^3(x)g'''(x) - U^2(x)U'''(x)g(x) = 0. \end{aligned} \quad (28)$$

Using Eq.(24) together with the eigenvalues of the Schrödinger equation for the Hamiltonian and its adjoint, namely  $\mathcal{H}' |\xi_i\rangle = \mathcal{E}'_i |\xi_i\rangle$  and  $\langle \xi_j | \mathcal{H}'^\dagger = \langle \xi_j | \mathcal{E}'_j^*$ , where  $|\xi_q\rangle \in \mathfrak{H}$  ( $q = i, j$ ), and then multiplying them by  $\tilde{\eta}$  on the left- and right-hand sides, respectively, we can easily obtain due to Eq.(24), on subtracting, that any two eigenvectors  $|\xi_i\rangle$  and  $|\xi_j\rangle$  satisfy

$$\begin{aligned}
\langle \xi_j | (\mathcal{H}'^\dagger \tilde{\eta} - \tilde{\eta} \mathcal{H}') |\xi_i\rangle &= \langle \xi_j | (\mathcal{E}'_j^* \tilde{\eta} - \mathcal{E}'_i \tilde{\eta}) |\xi_i\rangle \\
&= (\mathcal{E}'_j^* - \mathcal{E}'_i) \langle \xi_j | \tilde{\eta} |\xi_i\rangle \\
&= (\mathcal{E}'_j^* - \mathcal{E}'_i) \langle \xi_j \parallel \xi_i \rangle_{\tilde{\eta}} \\
&\equiv 0,
\end{aligned} \tag{29}$$

where  $\langle \xi_j \parallel \xi_i \rangle_{\tilde{\eta}} \equiv \langle \xi_j | \tilde{\eta} |\xi_i\rangle$  is the Hermitian indefinite inner product of the Hilbert space  $\mathfrak{H}$  defined by  $\tilde{\eta}$  [9,11]. According to the proposition 2 in Ref.[9], a direct implication of Eq.(29) has the following properties

- (i) The eigenvectors with non-real eigenvalues have a vanishing  $\eta$ -norm, i.e.  $\mathcal{E}'_i \notin \mathbb{R}$  implies that  $\| |\xi_i\rangle \|_{\tilde{\eta}}^2 = \langle \xi_i \parallel \xi_i \rangle_{\tilde{\eta}} = 0$ .
- (ii) Any two eigenvectors are  $\eta$ -orthogonal *unless* their eigenvalues are complex conjugates, i.e.  $\mathcal{E}'_i \neq \mathcal{E}'_j^*$  implies that  $\langle \xi_i \parallel \xi_j \rangle_{\tilde{\eta}} = 0$ .

The inner product  $\langle \cdot \parallel \cdot \rangle_{\tilde{\eta}}$  is generally positive-definite, i.e.  $\langle \cdot \parallel \cdot \rangle_{\tilde{\eta}} > 0$ . Thus, the Hilbert space equipped with this inner product may be identified as the physical Hilbert space  $\mathfrak{H}_{\text{phys}}$  [1-3]. Therefore, according to Eq.(29), it is obvious that  $\mathcal{E}' = \mathcal{E}'^*$ . Hence, the eigenvalue  $\mathcal{E}'$  is real, i.e.  $\mathcal{E}'_{\text{Im}} = 0$ . In terms of these,  $\eta$ -orthogonality suggests that the eigenvector (wave-function), here  $\xi(x)$ , is related to  $\mathcal{H}'$  through the identity  $\tilde{\eta}\xi(x) = 0$  [14], i.e.

$$\tilde{\mathcal{D}}\xi(x) = 0, \tag{30}$$

and keeping in mind Eq.(14), and after integration, we obtain the ground-

state wavefunction (not necessarily normalizable)

$$\begin{aligned}
\xi(x) &= \Lambda(x) \psi(x) \\
&= \exp \left[ i \int^x dy \frac{A(y)}{U(y)} \right] \psi(x) \\
&\propto \exp \left[ - \int^x dy \frac{f(y)}{U(y)} - i \int^x dy \frac{g(y) - a(y)}{U(y)} \right], \quad (31)
\end{aligned}$$

where  $\psi(x)$  is the ground-state wavefunction when the restriction  $A(x) = 0$  holds. Then  $\xi(x)$ , as for  $\tilde{\mathcal{D}}$ , is also subjected to a gauge transformation in the manner of  $\psi(x) \rightarrow \xi(x) = \Lambda(x) \psi(x)$ .

In these settings, letting  $\tilde{\mathcal{D}}$  acts on both sides of (31), we obtain

$$\begin{aligned}
\tilde{\mathcal{D}}\xi(x) &\equiv [U(x) \partial_x - iA(x) + \phi(x)] \Lambda(x) \psi(x) \\
&= U(x) \Lambda'(x) \psi(x) + U(x) \Lambda(x) \psi'(x) - iA(x) \Lambda(x) \psi(x) \\
&\quad + \phi(x) \Lambda(x) \psi(x) \\
&= \Lambda(x) [U(x) \partial_x + \phi(x)] \psi(x) \\
&\implies \mathcal{D}\psi(x) = 0, \quad (32)
\end{aligned}$$

where  $\Lambda'(x) = i \frac{A(x)}{U(x)} \Lambda(x)$ . That means that the wavefunctions thus obtained can be deduced either by  $\tilde{\mathcal{D}}\xi(x) = 0$  or by  $\mathcal{D}\psi(x) = 0$ .

In the remainder of the article, we write  $\mathcal{E}$  instead of  $\mathcal{E}'$ . Now, using the Schrödinger equation  $\mathcal{H}'\xi(x) = \mathcal{E}\xi(x)$ , with  $\mathcal{H}'$  given in Eq.(20),  $\xi(x)$  in Eq.(31) and  $\mathcal{E} = \mathcal{E}_{\text{Re}} + i\mathcal{E}_{\text{Im}}$ , we end up by relating  $f(x)$  to  $g(x)$  and  $U(x)$  through

$$f(x) = \frac{U'(x) g(x) - U(x) g'(x)}{2g(x)}, \quad (33)$$

where for the sake of simplicity we considere  $\delta \equiv \mathcal{E}_{\text{Re}}$ . Hence, it becomes clear that  $g(x)$  is our generating function leading to identify the function  $f(x)$ , and then the potential  $V(x)$ .

This in turn leads to the following question. Is (33) the equation connecting  $f(x)$  to the generating function  $g(x)$ ? The answer to this question amounts to check for the satisfaction of Eq.(28). It is then straightforward,

after a long calculation, to be convinced that  $f(x)$ , as defined in (33), is a farfetched function (solution).

In order to deal with position-dependent mass, we introduce the auxiliary function defined by the mapping  $\mu(x) \equiv \int^x \frac{dy}{U(y)}$ , where  $\mu(x)$  is a dimensionless mass integral which will appear frequently in subsequent developments. The function  $f(x)$  can be written as

$$f(x) = -\frac{g'(x)}{2\mu'(x)g(x)} - \frac{\mu''(x)}{2\mu'^2(x)}. \quad (34)$$

and the potential  $V(x)$  acquires the form

$$\begin{aligned} V_{\text{eff}}(x) - \mathcal{E}_{\text{Re}} = & -g^2(x) - \frac{g'^2(x)}{4g^2(x)\mu'^2(x)} + \frac{g''(x)}{2g(x)\mu'^2(x)} - \frac{g'(x)\mu''(x)}{2g(x)\mu'^3(x)} \\ & - 2i\frac{g'(x)}{\mu'(x)}, \end{aligned} \quad (35)$$

where  $V_{\text{eff}}(x)$  is called the effective potential and is related to  $V(x)$  by

$$V(x) = V_{\text{eff}}(x) - \mathcal{V}_\mu(x), \quad (36)$$

with

$$\mathcal{V}_\mu(x) = \frac{\mu'''(x)}{\mu'^3(x)} - \frac{5}{4} \frac{\mu''^2(x)}{\mu'^4(x)}. \quad (37)$$

## 4 Effective potentials and corresponding wavefunctions

The strategy to determine both effective potentials and ground-state wavefunctions is as follows. As  $g(x)$  is a generating function, all expressions depend on it. We may choose various generating functions  $g(x)$  and obtain all others expressions such as  $f(x)$ ,  $V_{\text{eff}}(x)$  and  $\tilde{\eta}$ . Knowing  $f(x)$  and  $g(x)$ , the proper ground-state wavefunctions can be found from Eq.(32), i.e. without the gauge-term. Without giving the details of our calculation which are straightforward, we present the results of various expressions in standard form.

## 4.1 $3D$ –Harmonic oscillator potential

$$g(x) = \alpha\mu(x), \quad (38.a)$$

$$f(x) = -\frac{1}{2\mu(x)} - \frac{\mu''(x)}{2\mu'^2(x)}, \quad (38.b)$$

$$V_{\text{HO}}(x) = -\alpha^2\mu^2(x) - \frac{1}{4\mu^2(x)} - 2i\alpha, \quad (38.c)$$

$$\psi_{\text{HO}}^{(0)}(x) \propto \frac{\sqrt{\mu(x)}}{U(x)} \exp\left[-\frac{i\alpha}{2}\mu^2(x)\right]. \quad (38.d)$$

## 4.2 Morse potential

$$g(x) = \exp[-\alpha\mu(x)], \quad (39.a)$$

$$f(x) = \frac{\alpha}{2} - \frac{\mu''(x)}{2\mu'^2(x)}, \quad (39.b)$$

$$V_{\text{M}}(x) = -\exp[-2\alpha\mu(x)] + 2i\alpha \exp[-\alpha\mu(x)] + \frac{\alpha^2}{4}, \quad (39.c)$$

$$\psi_{\text{M}}^{(0)}(x) \propto \frac{1}{U(x)} e^{-\frac{\alpha}{2}\mu(x)} \exp\left[\frac{2i}{\alpha} e^{-\alpha\mu(x)}(x)\right] \quad (39.d)$$

## 4.3 Scarf II potential

$$g(x) = \text{sech}[\alpha\mu(x)], \quad (40.a)$$

$$f(x) = \frac{\alpha}{2} \tanh[\alpha\mu(x)] - \frac{\mu''(x)}{2\mu'^2(x)}, \quad (40.b)$$

$$V_{\text{Sc}}(x) = -\left(1 + \frac{3\alpha^2}{4}\right) \text{sech}^2[\alpha\mu(x)] \\ + 2i\alpha \text{sech}[\alpha\mu(x)] \tanh[\alpha\mu(x)] + \frac{\alpha^2}{4}, \quad (40.c)$$

$$\psi_{\text{Sc}}^{(0)}(x) \propto \frac{1}{U(x) \sqrt{\cosh[\alpha\mu(x)]}} \exp\left[-\frac{i}{\alpha} \arctan \tanh \frac{\alpha}{2}\mu(x)\right] \quad (40.d)$$

#### 4.4 Generalized Pöschl-Teller potential

$$g(x) = \operatorname{cosech} [\alpha \mu(x)], \quad (41.a)$$

$$f(x) = \frac{\alpha}{2} \coth [\alpha \mu(x)] - \frac{\mu''(x)}{\mu'^2(x)}, \quad (41.b)$$

$$V_{\text{GPT}}(x) = - \left( 1 - \frac{3\alpha^2}{4} \right) \operatorname{cosech}^2 [\alpha \mu(x)] + 2i\alpha \operatorname{cosech} [\alpha \mu(x)] \coth [\alpha \mu(x)] + \frac{\alpha^2}{4}, \quad (41.c)$$

$$\psi_{\text{GPT}}^{(0)}(x) \propto \frac{1}{U(x) \sqrt{\sinh [\alpha \mu(x)]}} \tanh^{-\frac{2i}{\alpha}} \left[ \frac{\alpha \mu(x)}{2} \right]. \quad (41.d)$$

#### 4.5 Pöschl-Teller potential

$$g(x) = \operatorname{sech} [\alpha \mu(x)] \operatorname{cosech} [\alpha \mu(x)], \quad (42.a)$$

$$f(x) = \alpha \coth [2\alpha \mu(x)] - \frac{\mu''(x)}{2\mu'^2(x)}, \quad (42.b)$$

$$V_{\text{PT}}(x) = \left( \frac{3\alpha^2}{4} - 1 + 2i\alpha \right) \operatorname{cosech}^2 [\alpha \mu(x)] - \left( \frac{3\alpha^2}{4} - 1 - 2i\alpha \right) \operatorname{sech}^2 [\alpha \mu(x)] + \alpha^2, \quad (42.c)$$

$$\psi_{\text{PT}}^{(0)}(x) \propto \frac{1}{U(x) \sqrt{\sinh [2\alpha \mu(x)]}} \tanh^{-\frac{2i}{\alpha}} [\alpha \mu(x)]. \quad (42.d)$$

The above models are displayed in their usual forms and give quite well-known exact solvable non-Hermitian effective potentials as well as their accompanying ground-state wavefunctions. The first one represents a generalized  $\eta$ -weak-pseudo-Hermitian  $3D$ -harmonic oscillator. The second model corresponds to the non- $\mathcal{PT}$ -symmetric Morse potential and is already obtained by [22,23], where the  $\gamma = b_R$  constraint is considered therein, using  $\mathfrak{sl}(2, \mathbb{C})$  potential algebra as a complex Lie algebra by a simple complexification of the coordinates in a group theoretical point of view

and also in [24], labelled LIII according to Lévai [25], once a substitution  $b \rightarrow ib$  is made therein. The remainder models belong to so called PI class [25] which contains five individual potentials. The third model represents a generalized  $\eta$ -weak-pseudo-Hermitian  $\mathcal{PT}$ -symmetric Scarf II Potential, labelled  $\text{PI}_1$ , which is established in [22,23,24] with the same constraints quoted above. Finally, the two last models represent, respectively, a generalized  $\eta$ -weak-pseudo-Hermitian generalized Pöschl-Teller ( $\text{PI}_2$ ) and a generalized  $\eta$ -weak-pseudo-Hermitian Pöschl-Teller ( $\text{PI}_5$ ) potentials and are already established, respectively, in [22,23,24] and [24].

## 5 Conclusion

A well-known class of non-Hermitian Hamiltonians endowed with position-dependent mass are generated as a by-product of a generalized  $\eta$ -weak-pseudo-Hermiticity thanks to a shift on the momentum  $p$  of the type  $p \rightarrow p - \frac{A(x)}{U(x)}$ , and which allows to avoid the Hermitian invertible linear operator  $\eta$  for the benefit of  $\tilde{\eta}$ . We show that, being different from the realization of Ref.[13], there is no inconsistency to generate a well-known class of non-Hermitian Hamiltonians if the last shift is used, leading then to consider that  $\tilde{\mathcal{D}}$  may be looked upon as a gauge-transformed version of  $\mathcal{D}$  and depending essentially on the function  $A(x)$ , i.e.  $\delta\mathcal{D} \equiv \tilde{\mathcal{D}} - \mathcal{D} = -iA(x)$ . As a consequence of this, the wavefunction  $\xi(x)$  is also subjected to a gauge transformation in the manner  $\psi(x) \rightarrow \xi(x) = \Lambda(x)\psi(x)$ , with  $\Lambda(x) = \exp\left[i \int^x dy \frac{A(y)}{U(y)}\right]$  and where  $\psi(x)$  is the ground-state wavefunction when the  $A(x) = 0$  constraint holds.

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